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NUMERICAL AND ANALYTICAL APPROXIMATION FOR SOLUTION OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATION

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Abstract

At the present paper we workout an exact solution for first order non-linear integro-differential equation subject to initial condition can be converted into second order non linear differential equation by taking suitable substitution for integral part and also ideas of error function to make the series solution become converges.

Key Words and Phrases : *Picard's method of successive approximations, Asymptotic approximations, Nonlinear integro-differential equation, Pade approximation, and Error functions.*

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1. Introduction

By a Numerical-Analytic method [3, 11], we mean application of a suitable numerical method like method of asymptotic approximation [4] in analytic methods like any method of successive approximations [12] or any perturbation method or any Homotopy analysis method [9] for solving any problem of nonlinear dynamics [6]. We also seek for methods capable of producing a good approximate solution in one or two steps with the help of an asymptotic expansion suitable for application of asymptotic approximation. The methods are capable of fitting all the initial conditions as well as any other additional conditions on the nature of the solution.

2. Example

In the present study we consider the first order nonlinear Integro-differential equation [7, 8, 13, 14] of the following type:

$$\frac{du}{dt} = (-2k^2t)\ u \quad + \quad \frac{\epsilon}{[1 - \int_0^t u(x)dx]^2} \tag{1}$$

$$u(0) \ = \ u_0 = \sqrt{2}\ e^{-k\epsilon}$$

Note:

$$i)\ \frac{du}{dt} \ = \ (-2k^2t)\ u, \quad \text{has exact solution} \quad u(t) = Ae^{-k^2t^2}$$

$$ii)\ \frac{du}{dt} \ = \ \frac{\epsilon}{[1 - \int_0^t u(x)dx]^2}, \ u(0) = \sqrt{2}, \quad \text{also has exact solution } u(t) = \frac{\sqrt{2}}{\left(1 - \frac{3}{\sqrt{2}}t\right)^{\frac{1}{3}}}$$

putting $y(t) = \int_0^t u(x)dx$, it follows that $y^{'}(t) = u(t)$. then equ.(1) can be rewrite into a second order differential equation of the form

$$y^{''}(t) = (-2k^2t)\ y^{'} + \frac{\epsilon}{[1 - y(x)]^2} \tag{2}$$

subject to the initial conditions $y(0) = 0$, *and* $y^{'}(0) = \sqrt{2}\ e^{-k\epsilon}$.

The equ.(2) is equivalent to the following integral equation:

$$y(t) \ = \ \sqrt{2}\ e^{-k\epsilon}\ t + (-2k^2)\int_0^t (t - x)xy^{'}(x)dx + \epsilon \int_0^t (t - x)\frac{1}{[1 - y(x)]^2}dx \tag{3}$$

In the reference [10, 5], it is clearly stated that

$$\int_0^t (t - x)xy^{'}(x) \ = \ -\int_0^t (t - 2x)y(x)dx$$

Using the above idea, equ.(3) can be rewrite as follows

$$y(t) = \sqrt{2}\, e^{-k\epsilon}\, t + (2k^2)\int_0^t (t-2x)y(x)dx + \epsilon \int_0^t (t-x)\frac{1}{[1-y(x)]^2}dx \tag{4}$$

$$Let\ \ y(t) \ = \ \sum_{n=0}^{\infty} y_n(t) = y_0(t) + y_1(t) + y_2(t) + y_3(t) + \cdots + y_n(t) + \cdots$$

be series solution of (4), and choose $y_0(t) = 0$.

We apply the Binomial expansion [1, 2] for the series $(1 - y(x))^2$ one can write,

$$\begin{aligned}
\frac{1}{(1-y(x))^2} &= 1 + 2y(x) + 3(y(x))^2 + \cdots + (n+1)(y(x))^n + \cdots \\
&= 1 + 2(y_1 + y_2 + y_3 + \cdots) + 3(y_1 + y_2 + y_3 + \cdots)^2 + \cdots \\
&= 1 + 2y_1 + (2y_2 + 3y_1^2) + (2y_3 + 6y_1y_2 + 4y_1^3) + \cdots
\end{aligned} \tag{5}$$

On substituting equ.(5) in equ.(4) it gives,

$$\begin{aligned}
\sum_{n=0}^{\infty} y_n(t) &= (0 + \sqrt{2}\, e^{-k\epsilon}t) + (2k^2)\int_0^t (t-2x)(y_0 + y_1 + y_2 + y_3 + \cdots)\ dx \\
&\quad + \epsilon \int_0^t (t-x)[1 + 2y_1 + (2y_2 + 3y_1^2) + (3y_3 + 6y_1y_2 + 4y_1^3) + \cdots]\ dx
\end{aligned}$$

The components $y_0(t), y_1(t) \cdots y_n(t) \cdots$ of the unknown function $y(t)$ can be complectly determined in recurrence manner if we set, [12]

$$\begin{aligned}
y_0(t) &= 0 \\
y_1(t) &= \sqrt{2}\, e^{-k\epsilon}t + \epsilon \int_0^t (t-x)1dx \\
&= \sqrt{2}\, e^{-k\epsilon}t + \epsilon\frac{t^2}{2} \\
y_2(t) &= (2k^2)\int_0^t (t-2x)y_1(x)dx + \epsilon \int_0^t (t-x)(2y_1(x))dx \\
&= 2(\epsilon - k^2)\sqrt{2}\, e^{-k\epsilon}\left(\frac{t^3}{6}\right) + \epsilon^2(1-2k^2)\left(\frac{t^4}{12}\right) \\
y_3(t) &= (2k^2)\int_0^t (t-2x)y_2(x)dx + \epsilon \int_0^t (t-x)(2y_2(x) + 3y_1^2(x))dx
\end{aligned}$$

$$
\begin{aligned}
= \quad & 6\epsilon\ e^{-2k\epsilon}\frac{t^4}{12} + \left[-2k^2(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon} + \frac{2\epsilon(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon}}{3}\right]\frac{t^5}{20} \\
+ \quad & \left[\frac{-2k^2\epsilon^2(1-2k^2)}{3} + \frac{\epsilon^3(1-2k^2)}{6} + \frac{3\epsilon^3}{4}\right]\frac{t^6}{30}.
\end{aligned}
$$

$$
\begin{aligned}
y_4(t) =& (2k^2)\int_0^t (t-2x)y_3(x)dx + \epsilon\ \int_0^t (t-x)(2y_3(x) + 6y_1(x)y_2(x) + 4y_1^3(x))dx \\
=& \left[4\epsilon 2^{\frac{3}{2}} e^{-3k\epsilon}\right]\frac{t^5}{20} + \left[-4\epsilon\ k^2 e^{-2k\epsilon} + \epsilon^2 e^{-k\epsilon} + 4\epsilon e^{-2k\epsilon} + 12\epsilon^2 e^{-2k\epsilon}\right]\frac{t^6}{30} \\
&+ \left[\left(k^4(\epsilon - k^2) - \frac{k^2\epsilon(\epsilon - k^2)}{3} - \frac{k^2\epsilon}{2}\right)\sqrt{2}e^{-k\epsilon}\right. \\
&+ \left(\frac{-2k^2\epsilon(\epsilon - k^2)}{10} + \frac{\epsilon^2(\epsilon - k^2)}{30} + \frac{\epsilon^2}{10}\right) \\
&+ \left.\left(\frac{\epsilon^3(1-2k^2)}{2} - \epsilon^2(\epsilon - k^2)\right)\sqrt{2}e^{-2k\epsilon} + 3\epsilon^3\sqrt{2}e^{-k\epsilon}\right]\frac{t^7}{42} \\
&+ \left[\frac{4k^4\epsilon^2(1-2k^2)}{15} + \frac{2k^2\epsilon^3(1-2k^2)}{5} - \frac{3k^2\epsilon^2}{10}\right. \\
&+ \left.\frac{-k^2\epsilon^2(1-2k^2)}{15} + \frac{\epsilon^3(1-2k^2)}{30} + \frac{\epsilon^2}{40} + \frac{\epsilon^4(1-2k^2)}{4} + \frac{\epsilon^4}{2}\right]\frac{t^8}{56}.
\end{aligned}
$$

The series solution is

$$
\begin{aligned}
y(t) =& y_0(t) + y_1(t) + y_2(t) + y_3(t) + y_4(t) + \cdots \\
=& \sqrt{2}\ e^{-k\epsilon}t + \epsilon\frac{t^2}{2} + \left[2(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon}\right]\frac{t^3}{6} + \left[\epsilon^2(1-2k^2) + 6\epsilon\ e^{-2k\epsilon}\right]\frac{t^4}{12} \\
&+ \left[-2k^2(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon} + \frac{2}{3}\epsilon(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon} + 3\epsilon^2\sqrt{2}\ e^{-k\epsilon} + 4\epsilon 2^{\frac{3}{2}} e^{-3k\epsilon}\right]\frac{t^5}{20} \\
&+ \left[\frac{-2k^2\epsilon^2(1-2k^2)}{3} + \frac{\epsilon^3(1-2k^2)}{6} + \frac{3\epsilon^3}{4}\right. \\
&\qquad\qquad \left. -4\epsilon\ k^2 e^{-2k\epsilon} + \epsilon^2 e^{-k\epsilon} + 4\epsilon e^{-2k\epsilon} + 12\epsilon^2 e^{-2k\epsilon}\right]\frac{t^6}{30} \\
&+ \left[\left(k^4(\epsilon - k^2) - \frac{k^2\epsilon(\epsilon - k^2)}{3} - \frac{k^2\epsilon}{2}\right)\sqrt{2}e^{-k\epsilon} - \frac{k^2\epsilon(\epsilon - k^2)}{5} + \frac{\epsilon^2(\epsilon - k^2)}{15} + \frac{\epsilon^2}{10}\right. \\
&+ \left.\left(\frac{\epsilon^3(1-2k^2)}{2} - \epsilon^2(\epsilon - k^2)\right)\sqrt{2}e^{-2k\epsilon} + 3\epsilon^3\sqrt{2}\ e^{-k\epsilon}\right]\frac{t^7}{42} \\
&+ \left[\frac{4k^4\epsilon^2(1-2k^2)}{15} - \frac{2k^2\epsilon^3(1-2k^2)}{5} - \frac{3k^2\epsilon^2}{10} - \frac{2k^2\epsilon^3(1-2k^2)}{15} + \frac{\epsilon^4(1-2k^2)}{15}\right. \\
&+ \left.\frac{\epsilon^3}{20} + \frac{\epsilon^4(1-2k^2)}{4} + \frac{\epsilon^4}{2}\right]\frac{t^8}{56}.
\end{aligned}
$$

When $\epsilon = 0$, it gives,

$$\begin{aligned} y(t) &= \sqrt{2}\left[t - \frac{k^2t^3}{3} + \frac{k^4t^5}{10} - + \cdots\right] \\ &= \sqrt{\frac{\pi}{2}}\ erf(kt). \end{aligned}$$

Now,

$$\begin{aligned}
u(t) =& y^{'}(t) \\
=& \sqrt{2}\ e^{-k\epsilon} + \epsilon t + \Big[(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon}\Big]\, t^2 + \Big[\epsilon^2(1-2k^2) + 6\epsilon\ e^{-2k\epsilon}\Big]\,\frac{t^3}{3} \\
&+ \bigg[-2k^2(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon} + \frac{2}{3}\epsilon(\epsilon - k^2)\sqrt{2}\ e^{-k\epsilon} + 3\epsilon^2\sqrt{2}\ e^{-k\epsilon} + 4\epsilon 2^{\frac{3}{2}} e^{-3k\epsilon}\bigg]\,\frac{t^4}{4} \\
&+ \bigg[\frac{-2k^2\epsilon^2(1-2k^2)}{3} + \frac{\epsilon^3(1-2k^2)}{6} + \frac{3\epsilon^3}{4} \\
&\qquad\qquad -4\epsilon\ k^2e^{-2k\epsilon} + \epsilon^2e^{-k\epsilon} + 4\epsilon e^{-2k\epsilon} + 12\epsilon^2e^{-2k\epsilon}\ \Bigg]\,\frac{t^5}{5} \\
&+ \bigg[\bigg(k^4(\epsilon - k^2) - \frac{k^2\epsilon(\epsilon - k^2)}{3} - \frac{k^2\epsilon}{2}\bigg)\ \sqrt{2}e^{-k\epsilon} - \frac{k^2\epsilon(\epsilon - k^2)}{5} + \frac{\epsilon^2(\epsilon - k^2)}{15} + \frac{\epsilon^2}{10} \\
&+ \bigg(\frac{\epsilon^3(1-2k^2)}{2} - \epsilon^2(\epsilon - k^2)\bigg)\ \sqrt{2}e^{-2k\epsilon} + 3\epsilon^3\sqrt{2}\ e^{-k\epsilon}\bigg]\,\frac{t^6}{6} \\
&+ \bigg[\frac{4k^4\epsilon^2(1-2k^2)}{15} - \frac{2k^2\epsilon^3(1-2k^2)}{5} - \frac{3k^2\epsilon^2}{10} - \frac{2k^2\epsilon^3(1-2k^2)}{15} + \frac{\epsilon^4(1-2k^2)}{15} \\
&+ \frac{\epsilon^3}{20} + \frac{\epsilon^4(1-2k^2)}{4} + \frac{\epsilon^4}{2}\bigg]\,\frac{t^7}{7}.
\end{aligned}$$

When $\epsilon = 1$ *and* $k = 0$ it follows that $u(t)$ becomes

$$\begin{aligned} u(t) &= \sqrt{2}\left[1 + \frac{t}{\sqrt{2}} + t^2 + \frac{7}{3}t^3 + \cdots\right] \\ &= \sqrt{2}\left[1 + \frac{\frac{1}{3}}{1!}\left(\frac{3}{\sqrt{2}}\ t\right) + \frac{\frac{1}{3}\frac{3}{4}}{2!}\left(\frac{3}{\sqrt{2}}\ t\right)^2 + \cdots\right] \\ &= \frac{\sqrt{2}}{\left(1 - \frac{3}{\sqrt{2}}t\right)^{\frac{1}{3}}}, \end{aligned}$$

which is an exact solution of equ.(2).

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